Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a) $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr^2

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\begin{aligned}\int_0^8 R(t)\,dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters}\end{aligned}$$

This is an overestimate since R is a decreasing function.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $$\begin{aligned}\text{Total} &\approx 50000 + \int_0^8 W(t)\,dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters}\end{aligned}$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

Therefore, the Intermediate Value Theorem guarantees at least one time t, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

$$R'(2) \approx \frac{R(3) - R(1)}{3-1} = \frac{950-1190}{2}$$

$$R'(2) \approx -120\ L/hr^2$$

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(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

$$\int_0^8 R(t)\,dt \approx L_4 = (1)(1340) + (2)(1190) + (3)(950) + (2)(740)$$

$$\int_0^8 R(t)\,dt \approx 8{,}050\ L$$

This is an overestimate because we are taking the Left Riemann sum of a decreasing function. Therefore the left endpoints, are greater than the right endpoints in each subinterval.

Continue problem 1 on page 5.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

$t=0 \to 50,000$ L

Total Left $\approx 50,000 + \left(\int_0^8 W(t)\,dt - \int_0^8 R(t)\,dt\right)$

$\int_0^8 W(t)\,dt = 7836.19532455$

Total Left $\approx 50,000 + (7836.19532455 - 8050)$

$\boxed{\text{Total Left} \approx 49,786\text{ L}}$

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

$T(t) = W(t) - R(t)$

$T(0) = 2000 - 1340 = 600$

$T(8) = 81.524 - 700 = -618.47$

$\boxed{-618.47 < 0 < 600}$

When $T(t)=0$, the rate of water being pumped in to the tank is equal to the rate of water being pumped out. Because the function $T(t)$ is continuous and differentiable, by IVT there exists some t on the interval $[0, 8]$ where $T(t) = 0$.

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t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

$$R'(2) \approx \frac{950 - 1190}{3 - 1} = \boxed{-120 \text{ liters per hour}^2}$$

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

$1340 + (2)(1190) + (3)(950) + (2)(740) = 8050$

8050 liters is the total amount of water removed during the 8 hours. The left Riemann sum is an overestimate since $R(t)$ is decreasing and the graph is below the Riemann sum.

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Continue problem 1 on page 5.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

$$\int_0^8 (W(t) - R(t))\,dt$$

$$50{,}000 + \left(\int_0^8 (W(t))\,dt - 8050\right)$$

$$50{,}000 + (7836.1953 - 8050)$$

49786.7953 liter

49,787 liters of water is in the tank.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

No there is no rate that is the same as the rate at which water is being removed because $R'(t)$ is decreasing at a slower rate than $W'(t)$ so the rate at which $W'(t)$ is decreasing is much greater than $R'(t)$.

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$\frac{y_2-y_1}{x_2-x_1}$

x t (hours)	0	1	3	6	8
y $R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

$\frac{y_2-y_1}{x_2-x_1} = \frac{1190-1340}{1-0} = -150 \ \frac{L^2}{hr}$

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

$1(1340) + 2(1190) + 3(950) + 2(740) = 21370 \ \frac{\text{liters}}{\text{hour}}$

over estimate b/c left Riemann sum.

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Continue problem 1 on page 5.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

$50{,}000 + \int_0^8 w(t)\,dt - 21370 =$ 36466 Liters in the tank after 8 hours

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

yes b/c the ammount of water pumped out is increasing.

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Question 1

Overview

In this problem students were given a function W that models the rate, in liters per hour, at which water is pumped into a tank at time t hours. They were also given a function R that models the rate, in liters per hour, at which water is removed from the tank. W is defined as an exponential function on the interval $0 \le t \le 8,$ and R is differentiable and decreasing on $0 \le t \le 8.$ Selected values of $R(t)$ are given in a table. The amount of water in the tank, in liters, at time $t = 0$ is given. In part (a) students needed to estimate $R'(2)$ by calculating the value of an appropriate difference quotient based on the values in the table. Units of liters/hr^2 are required. In part (b) students needed to use a left Riemann sum approximation for $\int_0^8 R(t)\,dt$ to estimate the total amount of water removed from the tank during the interval $0 \le t \le 8.$ Students needed to use the appropriate function values from the table with the four subintervals $[0, 1],$ $[1, 3],$ $[3, 6],$ and $[6, 8].$ By applying the given information that R is decreasing, students needed to conclude that the left Riemann sum approximation is an overestimate. In part (c) students needed to estimate the total amount of water in the tank at time $t = 8.$ This required adding the amount of water in the tank at time $t = 0$ to the amount of water pumped into the tank during the interval $0 \le t \le 8,$ and then subtracting the overestimate found in part (b). The definite integral $\int_0^8 W(t)\,dt$ gives the amount of water pumped into the tank during the interval $0 \le t \le 8$ and is evaluated using the calculator. In part (d) students needed to apply the Intermediate Value Theorem to $W(t) - R(t).$ This theorem guarantees at least one time t on the interval $0 < t < 8$ for which $W(t) - R(t) = 0$ or $W(t) = R(t).$ For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

Sample: 1A
Score: 9

The response earned all 9 points.

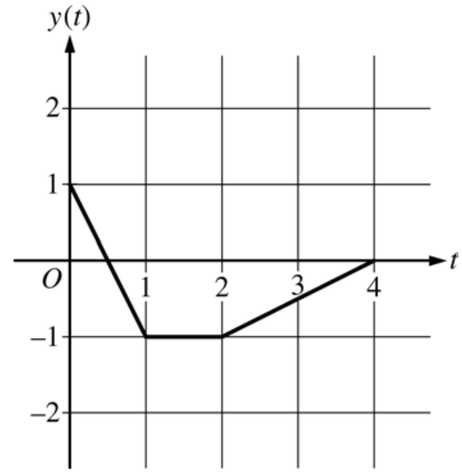
Sample: 1B
Score: 6

The response earned 6 points: 2 points in part (a), 3 points in part (b), 1 point in part (c), and no points in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student earned the first point for the correct definite integral. The student did not earn the second point for the estimate because of an arithmetic error. In part (d) the student earned no points.

Sample: 1C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student calculates an estimate using an incorrect interval, and the units are incorrect. In part (b) the student has a correct left Riemann sum and earned the first point. The student has an incorrect estimate and does not support the answer of an overestimate with a valid reason. In part (c) the student earned the first point for the definite integral. The estimate is consistent with the student's estimate from part (b), so the second point was earned. In part (d) the student earned no points.

Question 2



At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $x(3) = x(0) + \int_0^3 x'(t)\,dt = 5 + 9.377035 = 14.377$

$y(3) = -\frac{1}{2}$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) Slope $= \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$

1 : slope

(c) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969$ (or 9.968)

$2 : \begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

(d) Distance $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\,dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2}\,dt$

$= 2.237871 + 2.112003 = 4.350$ (or 4.349)

$3 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

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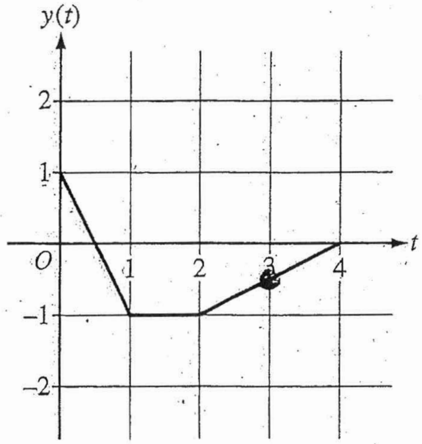

2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

$t=3 \qquad y = -0.5$

$\boxed{t=3 \quad (14.377, -0.5)}$

$$\int_0^3 (t^2 + \sin(3t^2))\,dt = x(3) - x(0)$$

$$5 + \int_0^3 (t^2 + \sin(3t^2))\,dt = 14.377$$

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

$t=3 \rightarrow$

$y = -0.5$

$x = 14.377$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{y(4) - y(2)}{4-2}}{3^2 + \sin(3(3^2))}$$

$$\frac{dy}{dx} = \frac{\frac{0+1}{2}}{9 + \sin 27} = \frac{1}{2} \cdot \frac{1}{9 + \sin 27} = \boxed{0.050}$$

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Continue problem 2 on page 7.

(c) Find the speed of the particle at $t = 3$.

$$\text{SPEED} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\text{SPEED} = \sqrt{(9 + \sin 27)^2 + \left(\frac{1}{2}\right)^2}$$

$$\text{SPEED @ } t=3 \Rightarrow \boxed{9.969 \text{ u/time}}$$

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$$TD = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$TD = \int_0^1 \sqrt{(t^2 + \sin(3t^2))^2 + (-2)^2} +$$

$$\int_1^2 \sqrt{(t^2 + \sin(3t^2))^2 + (0)^2}$$

$$\boxed{TD = 4.350 \text{ units}}$$

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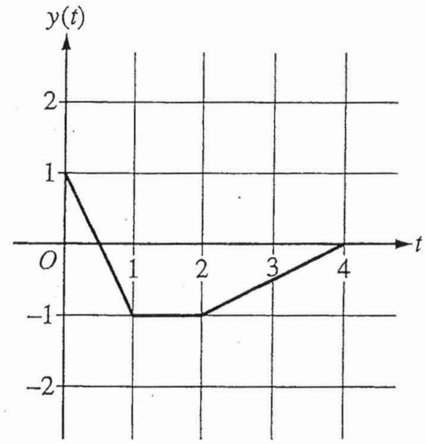

2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

$t = 3 \rightarrow y = -0.5$ from graph

$x = \int_0^3 (t^2 + \sin(3t^2))\,dt + 5$

$x = 14.377$

$(14.377, -0.5)$

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$\left.\frac{dy}{dt}\right|_{t=3} = \frac{0-(-1)}{4-2} = \frac{1}{2}$

$\left.\frac{dx}{dt}\right|_{x=3} = 3^2 + \sin(3 \cdot 3^2) = 9.956376$

$\frac{dy}{dx} = \frac{9.956376}{\frac{1}{2}} = 19.913$

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Continue problem 2 on page 7.

2 of 2
2B2

(c) Find the speed of the particle at $t = 3$.

$$s\Big|_{t=3} = \sqrt{\left(\frac{dx}{dt}\Big|_{t=3}\right)^2 + \left(\frac{dy}{dt}\Big|_{t=3}\right)^2}$$

$$= \sqrt{(1.956376)^2 + (5)^2}$$

$$= 4.969$$

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$$d = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$d = \int_0^1 \sqrt{(t + \sin(3t^2))^2 + (-2t+1)^2}\, dt + \int_1^2 \sqrt{(t+\sin(3t^2))^2 + (-1)^2}\, dt$$

$$d = 3.377$$

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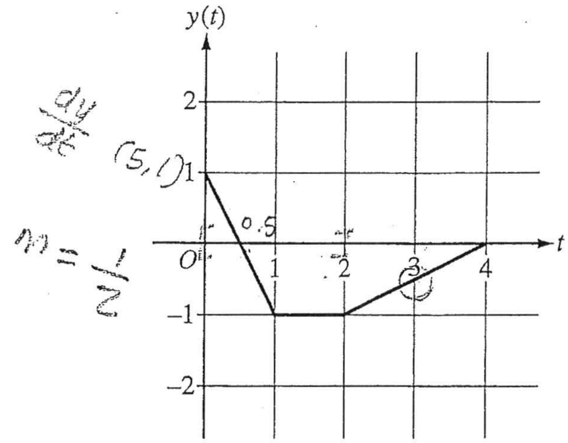

2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

$(5,1)$ $t = 0$

$$\frac{dx}{dt} = (3)^2 + \sin(27) = 9.956$$

$x = 5 + 9.956$ $\quad y = 1 + \frac{1}{2}$

$$\boxed{(14.956, 1.5)}$$

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

$$\frac{dx}{dt} = t^2 + \sin(3t^2) = 9 + \sin(27)$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{0.5}{9 + \sin 27} = \boxed{0.050}$$

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Continue problem 2 on page 7.

2 of 2
2C2

(c) Find the speed of the particle at $t = 3$.

$$\text{speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$$

$$= \sqrt{(9 + \sin 27)^2 + (0.5)^2}$$

$$= 9.969$$

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$$\int_0^2 (t^2 + \sin(3t^2)) + \int_0^2 y(t)$$

$$2.960 + (0.25 + 0.25 + 1)$$

$$2.960 + 1.50$$

$$= \boxed{3.460}$$

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Question 2

Overview

In this problem students were given information about the motion of a particle in the *xy*-plane. The position of the particle is defined as $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin\left(3t^2\right)$. The graph of y, consisting of three line segments, and the position of the particle at $t = 0$ were also given. In part (a) students needed to find the position of the particle at $t = 3$. Students were expected to evaluate $5 + \int_0^3 \left(t^2 + \sin\left(3t^2\right)\right) dt$ using the calculator to find $x(3)$. The value of $y(3)$ can be read from the graph. In part (b) students needed to compute the slope of the line tangent to the particle's path at $t = 3$ by evaluating $\frac{y'(3)}{x'(3)}$. The value for $y'(3)$ is found by computing the slope of the line segment from $t = 2$ to $t = 4$, and the value for $x'(3)$ is found using the calculator. In part (c) students needed to find the speed of the particle at $t = 3$. Students were expected to write the expression for speed, $\sqrt{(x'(3))^2 + (y'(3))^2}$, and use the calculator to compute this value. In part (d) students needed to find the total distance traveled from $t = 0$ to $t = 2$. Although the general formula for total distance traveled is $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$, students were expected to write the total distance traveled as the sum of two integrals since $y'(t) = -2$ from $t = 0$ to $t = 1$, and $y'(t) = 0$ from $t = 1$ to $t = 2$. The expression $\int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\, dt + \int_1^2 \sqrt{(x'(t))^2 + (0)^2}\, dt$ is evaluated using the calculator.

Sample: 2A
Score: 9

The response earned all 9 points.

Sample: 2B
Score: 6

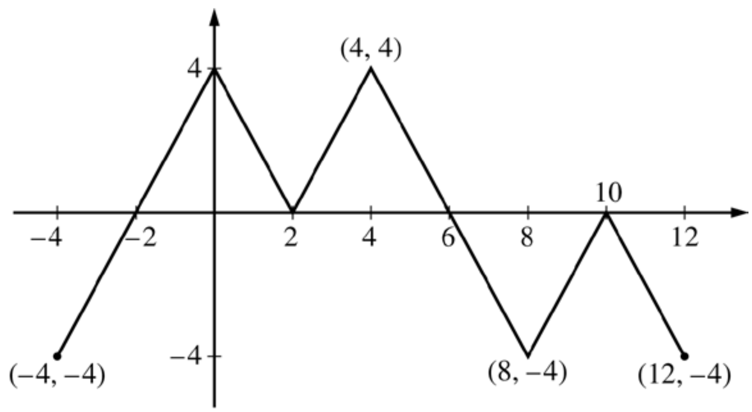
The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student calculates $\frac{dx}{dy}$ instead of $\frac{dy}{dx}$ and did not earn the point. In part (c) the student's work is correct. In part (d) the student presents a single integral for total distance traveled and earned the first point. The student attempts to present the total distance traveled as the sum of two integrals. The $\frac{dy}{dt}$ in each integrand is incorrect. The student did not earn any other points.

Sample: 2C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student evaluates $x'(3)$ instead of $x(3)$. The student was not eligible to earn any points. In parts (b) and (c) the student's work is correct. In part (d) the student does not present a correct integral expression for total distance traveled. The student was not eligible for any points.

Question 3

The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

---

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

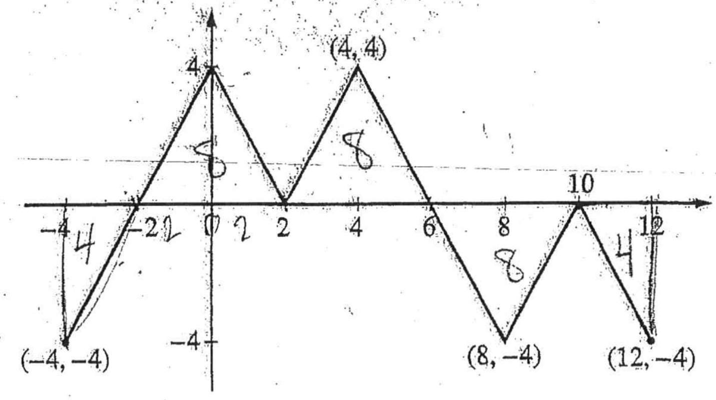
x	$g(x)$
−4	−4
−2	−8
6	8
12	−4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

$$4 : \begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \quad\ \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$$

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

$g'(x) = f(x)$

g does not have a relative minimum or maximum at $x=10$ because $g'(x) = f(x)$ does not change sign at this point

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

$g''(x) = f'(x)$

$f'(x) = g''(x)$ does change sign at $x=4$ so g does have a point of inflection at this point

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Continue problem 3 on page 15.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

$g'(x) = f(x) = 0 \quad x = -2 \quad x = 2$
$x = 6 \quad x = 10$

does not change sign at $x = 2$ and $x = 10$

x	$g(x)$
-4	$\int_2^{-4} f(t)\,dt = -8 + 4 = -4$
-2	$\int_2^{-2} f(t)\,dt = -8$
6	$\int_2^{6} f(t)\,dt = 8$
12	$\int_2^{12} f(t)\,dt = 8 - 8 - 4 = -4$

The absolute minimum value of g on the interval $-4 \le x \le 12$ is -8 and the absolute maximum value of g is 8

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$g(x) = \int_2^x f(t)\,dt \le 0$

$g(x) = 0$ at $x = 2$ and $x = 10$

$g(x) \le 0$ in the intervals $-4 \le x \le 2$ and $10 \le x \le 12$

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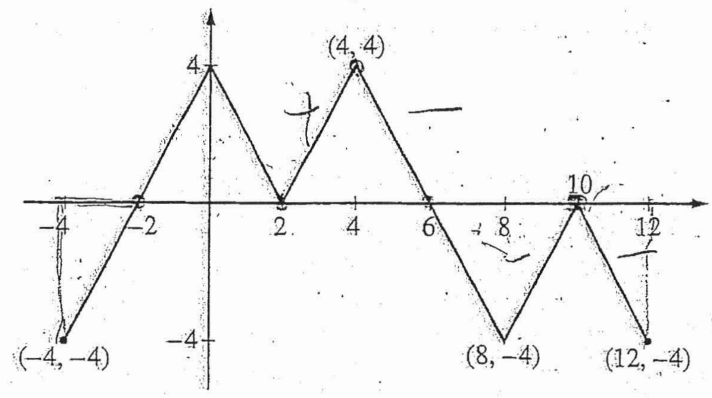

Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\, dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

$g'(x) = f(x)$

$g'(10) = f(10) = 0$

g is neither at $x=10$ bc $g'(x)$ does not change from pos to neg or neg to pos at $x = 10$.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

$g''(x) = f'(x)$

$f'(4) = 0$

g has a poi at $x=4$ bc $g''(4) = 0$ and $g''(x) > 0$ when $2 \le x < 4$ and $g''(x) < 0$ when $4 < x < 8$.

Continue problem 3 on page 15.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

$g'(x) = f(x) = 0$

$x = -2, 2, 6, 10$

x	$g(x)$
-4	$-\int_{-4}^{2} f(t)\,dt = -\left[(\frac{1}{2})(-4)(2) + (\frac{1}{2})(4)(4)\right] = -(-4+8) = -4$
-2	$-\int_{-2}^{2} f(t)\,dt = -\left[(\frac{1}{2})(4)(4)\right] = -8$
2	$\int_{2}^{2} f(t)\,dt = 0$
6	$\int_{2}^{6} f(t)\,dt = (\frac{1}{2})(4)(4) = 8$
10	$\int_{2}^{10} f(t)\,dt = 0$
12	$\int_{2}^{12} f(t)\,dt = (\frac{1}{2})(-4)(2) = -4$

abs max $\rightarrow x = 6$ abs min $\Rightarrow x = -2$

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$\int_{2}^{x} f(t)\,dt \le 0$

$\boxed{(10,12) \quad (-4,2)}$

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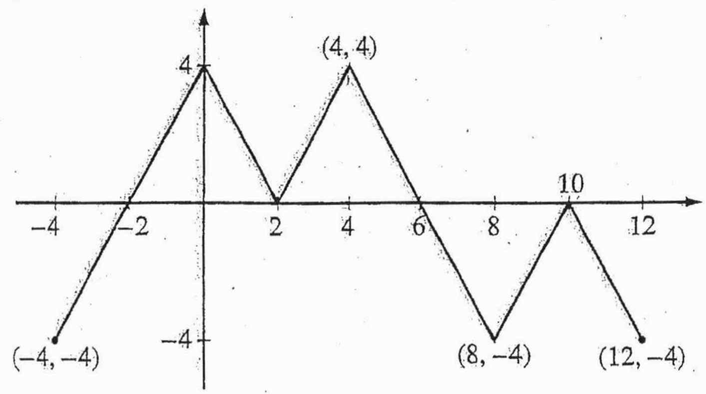
NO CALCULATOR ALLOWED



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

$g'(x) = f(x)$

Since $g'(x) = f(x)$, the graph of g has a relative maximum at $x=10$ because the graph of f increaces before $x = 10$ and decreaces after $x=10$ and $x=10$ is a critical point.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

$g''(x) = f'(x)$

Since $g''(x) = f'(x)$, the graph of g has an inflection point at $x=4$ because the graph of f increaces before $x=4$ and decreaces after $x=4$.

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Continue problem 3 on page 15.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

$g'(x) = f(x) = 0$

$x = -2, 2, 6, 10$

absolute maximum at $x = 10$ and absolute minimum at $x = 2$

. The absolute value for both extremas are 0 since

It is found by $g'(x) = f(x) = 0$.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$g(x)$ is decreasing when $g'(x)$ is ≤ 0 and $g'(x)$ is ≤ 0

since $g'(x) = f(x)$ and $g''(x) = f'(x)$, we know that

$6 < x < 10$ and $10 < x < 12$ are the only intervals

where both $g(x)$ and $g''(x)$, which is $f(x)$ and $f'(x)$,

are decreasing (having the same sign).

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Question 3

Overview

In this problem students were given the graph of f, a piecewise-linear function defined on the interval $[-4, 12]$. A second function g is defined by $g(x) = \int_2^x f(t)\,dt$. In part (a) students needed to determine whether g has a relative minimum, a relative maximum, or neither at $x = 10$, and justify their answer. Using the Fundamental Theorem of Calculus, students needed to recognize that $g'(x) = f(x)$ for all x in the interval $[-4, 12]$. Since $g'(10) = f(10) = 0$ and $f(x) \le 0$ for $[8, 12]$, the First Derivative Test may be applied to conclude that there is no relative extremum at $x = 10$. In part (b) students needed to determine whether the graph of g has a point of inflection at $x = 4$, and justify their answer. Since $g'(x) = f(x)$, the graph of g has a point of inflection at $x = 4$ because f changes from increasing to decreasing at $x = 4$. In part (c) students needed to find the absolute minimum value and the absolute maximum value of g on $[-4, 12]$. Since $g'(x) = f(x)$, students were expected to find relative extrema of g by identifying x-values where f changes sign. The absolute extrema occur either at the endpoints of the interval or at the relative extrema. By comparing the values of g at the four candidate x-values, students choose and justify the absolute extrema. Properties of the definite integral and the relation of the definite integral to accumulated area must be used to find the values of g. In part (d) students needed to find all intervals in $[-4, 12]$ for which $g(x) \le 0$. This part also required properties of the definite integral and the relation of the definite integral to accumulated area.

Sample: 3A
Score: 9

The response earned all 9 points. The student earned the $g'(x) = f(x)$ point in part (a). In part (a) the student earned the point with justification "$g'(x) = f(x)$ does not change sign at this point." In part (b) the student earned the point with justification "$f'(x) = g''(x)$ does change sign at $x = 4$." In part (c) the student identifies the absolute minimum and absolute maximum values with a candidates test that uses the necessary critical points. In part (d) the student gives the two correct closed intervals.

Sample: 3B
Score: 6

The response earned 6 points: 1 point for $g'(x) = f(x)$, 1 point in part (a), no points in part (b), 3 points in part (c), and 1 point in part (d). The student earned the $g'(x) = f(x)$ point in part (a). In part (a) the student earned the point with justification "$g'(x)$ does not change from pos to neg or neg to pos at $x = 10$." In part (b) the student gives the correct answer but includes an incorrect statement that $g''(4) = 0$. In part (c) the student earned the first 2 points. The student does not identify the absolute minimum as -8 or the absolute maximum as 8. The student earned 1 of the 2 answers with justification points. In part (d) the student does not include the endpoints of the intervals, so 1 point was earned.

Sample: 3C
Score: 3

The response earned 3 points: 1 point for $g'(x) = f(x)$, no points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). The student earned the $g'(x) = f(x)$ point in part (a). In part (a) the student has

Question 3 (continued)

an incorrect answer. In part (b) the student's work is correct. In part (c) the student earned the first point by identifying $x = -2$ and $x = 6$ in the second line. The student earned no other points. In part (d) the student has an incorrect interval $(6, 10)$ that has no values where $g(x) \le 0$.

Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$

2 : $\frac{d^2y}{dx^2}$ in terms of x and y

(b) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$

$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$

Thus, the graph of f has a relative maximum at the point $(-2, 8)$.

2 : conclusion with justification

(c) $\lim_{x\to -1}(g(x) - 2) = 0$ and $\lim_{x\to -1} 3(x+1)^2 = 0$

Using L'Hospital's Rule,

$\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right)$

$\lim_{x\to -1} g'(x) = 0$ and $\lim_{x\to -1} 6(x+1) = 0$

Using L'Hospital's Rule,

$\lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x\to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$

3 : $\begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$

$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$

2 : $\begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}$$

$$= 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$$

$$= \boxed{2x - \frac{1}{2}x^2 + \frac{1}{4}y}$$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} = 2(-2) - \frac{1}{2}(-2)^2 + \frac{1}{4}(8)$$

$$= -4 - 2 + 2 = -4$$

$$\left.\frac{dy}{dx}\right|_{(-2,8)} = (-2)^2 - \frac{1}{2}(8) =$$

$$= 4 - 4 = 0$$

f has a relative maximum at point $(-2,8)$ because $\frac{dy}{dx} = 0$ at $(-2,8)$ and $\frac{d^2y}{dx^2} < 0$ at $(-2,8)$.

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Continue problem 4 on page 17.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

$= \lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) ; \frac{0}{0}$

$g'(x)\Big|_{(-1,2)} = \frac{dy}{dx}\Big|_{(-1,2)} = (-1)^2 - \frac{1}{2}(2)$
$= 1-1 = 0$

$= \lim_{x\to -1}\left(\frac{g'(x)}{3\cdot 2(x+1)}\right) ; \frac{0}{0}$

$g''(x) = \frac{d^2y}{dx^2}\Big|_{(-1,2)} = 2(-1) - \frac{1}{2}\cdot 0$
$= -2$

$= \lim_{x\to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = \boxed{-\frac{1}{3}}$

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$\frac{dy}{dx}\Big|_{(0,2)} = 0 - \frac{1}{2}(2)$
$= -1$

$h(0.5) \approx 2 + (0.5)(-1)$
$= 2 - 0.5$
$= 1.5$

$\frac{dy}{dx}\Big|_{(0.5,1.5)}$
$= \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right)$
$= \frac{1}{4} - \frac{3}{4}$
$= -\frac{1}{2}$

$h(1) \approx 1.5 + (0.5)\left(-\frac{1}{2}\right)$
$= 1.5 - 0.25$
$= 1.25$

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NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$\frac{dy}{dx} = x^2 - \frac{1}{2}y$

$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}(x^2 - \frac{1}{2}y)$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$\frac{dy}{dx} = y' = f'(x$

$f'(x)$ 0 at -2

$\frac{dy}{dx} = x^2 - \frac{1}{2}y$

$\frac{dy}{dx} - x^2 = -\frac{1}{2}y$

$-2\frac{dy}{dx} + 2x^2 = y$

@ $x = -2$ $\frac{dy}{dx} = (-2)^2 - \frac{1}{2}(8)$

$= 4 - 4 = 0$

@ $x = -3$

@ $y = 0$

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Continue problem 4 on page 17.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

$g'' = y'' = \frac{d^2y}{dx^2}$

$\lim_{x \to -1} \left(\frac{g(x)-2}{3(x+1)^2} \right) \xrightarrow{\text{L'Hopital}} \lim_{x \to -1} \frac{g'(x)}{6(x+1)} \overset{\text{L'Hopital}}{\Rightarrow} \lim_{x \to -1} \frac{g''(x)}{6} = \frac{-2}{6} = -\frac{1}{3}$

$\frac{d^2y}{dx^2}(-1) = 2(-1) - \frac{1}{2}((-1)^2 - \frac{1}{2}(2))$
$= -2 - \cancel{\frac{1}{2}(1-1)}\,0$
$= -2$

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$h(0) = 2$
$h(0.5) = 2 + m\Delta x = 2 + (0.5)(-1) = 1.5$
$h(1) = 1.5 + m\Delta x = 2 + (0.5)(.5) = 2 - .25 = 1.75$

$y'(0) = 0 - \frac{1}{2}(2) = -1$
$y'(1.5) = (0.5)^2 - \frac{1}{2}(1.5)$
$= 0.25 - .75 = .5$

1.5
.5
.25

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\boxed{\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}}$$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$$\left.\frac{dy}{dx}\right|_{(-2,8)} = 4 - 4 = 0$$

the graph of f has a relative maximum at the point (-2, 8).

$$\left.\frac{dy}{dx}\right|_{(-3,8)} = 9 - 4 = 5 \leftarrow \text{pos} \therefore \text{inc}$$

$$\left.\frac{dy}{dx}\right|_{(-1,8)} = 1 - 4 = -3 \leftarrow \text{neg} \therefore \text{dec}$$

the graph goes from increasing to decreasing at (-2, 8), so the point is a relative maximum.

Continue problem 4 on page 17.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

$\left.\frac{dy}{dx}\right|_{(-1,2)} = 1 - 1 = 0$

$\lim_{x\to -1}\left(\frac{0-2}{3(x+1)^2}\right) = \lim_{x\to -1}\left(\frac{-2}{3(x+1)^2}\right) = \boxed{\frac{-2}{3}}$

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

step size $= 1/2$

$h(\frac{1}{2}) = h(0) + (0^2 - \frac{1}{2}(2))(\frac{1}{2})$

$h(\frac{1}{2}) = 2 + (-\frac{1}{2}) = \frac{3}{2}$

$h(1) \approx h(\frac{1}{2}) + ((\frac{1}{2})^2 - \frac{1}{2}(\frac{3}{2}))(\frac{1}{2})$

$h(1) \approx \frac{3}{2} + (\frac{1}{4} - \frac{3}{4})(\frac{1}{2}) \approx \frac{3}{2} + (-\frac{1}{2})(\frac{1}{2})$

$\frac{3}{2} - \frac{1}{4} = \frac{6}{4} - \frac{1}{4} = \frac{5}{4}$

$\boxed{h(1) \approx \frac{5}{4}}$

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Question 4

Overview

In this problem students were presented with a differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$. In part (a) students needed to find the second derivative $\frac{d^2y}{dx^2}$ in terms of x and y using implicit differentiation. In part (b) students were given that $y = f(x)$ is the particular solution to the differential equation whose graph passes through $(-2, 8)$. Students needed to use both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to determine if the graph of f has a relative minimum, a relative maximum, or neither at the point $(-2, 8)$, and justify their answer. In part (c) students were given that $y = g(x)$ is the particular solution to the differential equation with $g(-1) = 2$. Students needed to compute a limit. After indicating that L'Hospital's Rule is required, students were expected to apply L'Hospital's Rule twice. In part (d) students were given that $y = h(x)$ is the particular solution to the differential equation with $h(0) = 2$. Students needed to use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Sample: 4A
Score: 9

The response earned all 9 points.

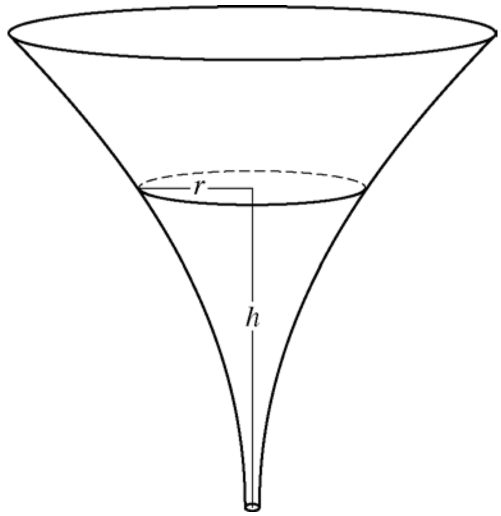
Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student correctly evaluates the first derivative at $(-2, 8)$. The student does not consider values of the second derivative and was not eligible for any points. Note that for this problem, use of the First Derivative Test to justify a correct conclusion is very difficult. The Second Derivative Test is much more appropriate here. In part (c) the student's work is correct. In part (d) the student finds a correct approximation for $h(0.5)$ and sets up a second iteration of Euler's method. The student earned the first point. The student makes an error in computing the second iteration and has an incorrect approximation.

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student differentiates correctly with respect to x and earned 1 point. The student does not present the second derivative in terms of x and y. In part (b) the student correctly evaluates the first derivative at $(-2, 8)$. The student does not consider values of the second derivative and was not eligible for any points. Note that for this problem, use of the First Derivative Test to justify a correct conclusion is very difficult. The Second Derivative Test is much more appropriate here. In part (c) the student makes an incorrect attempt to evaluate the limit and earned no points. In part (d) the student's work is correct.

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(a) Average radius $= \frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right) dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$

$$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60} \text{ in}$$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2 dh = \frac{\pi}{400}\int_0^{10}\left(9 + 6h^2 + h^4\right) dh$

$$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$$

$$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40} \text{ in}^3$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

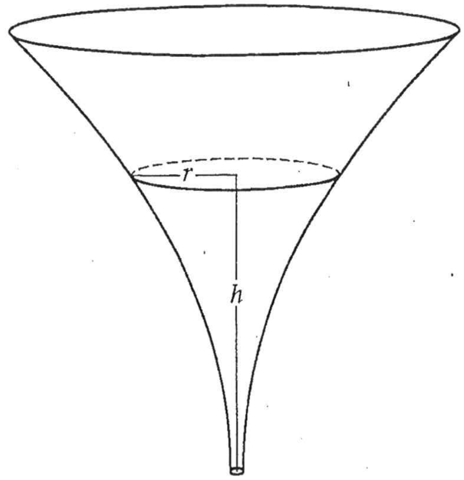
(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$

$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec}$$

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

$$r_{avg} = \frac{1}{200}\int_0^{10}(3+h^2)\,dh$$

$$= \frac{1}{200}\left[\left(3h + \frac{h^3}{3}\right)\Big|_0^{10}\right]$$

$$= \frac{1}{200}\left(30 + \frac{1000}{3}\right)$$

$$= \frac{109}{60} \text{ inches.}$$

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Continue problem 5 on page 19.

2 of 2

5A

(b) Find the volume of the funnel.

$$V = \frac{\pi}{400}\int_0^{10} (3+h^2)^2\,dh$$
$$= \frac{\pi}{400}\int_0^{10} (9+6h^2+h^4)\,dh$$
$$= \frac{\pi}{400}\left(9h+2h^3+\frac{h^5}{5}\right)\Big]_0^{10}$$
$$= \frac{\pi}{400}(90+2000+20000)$$
$$= \frac{2209\pi}{40}\ in^3$$

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$r = \frac{1}{20}(3+h^2)$$
$$\frac{dr}{dt} = \left(\frac{h}{10}\right)\frac{dh}{dt}$$
$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$$
$$\frac{dh}{dt} = -\frac{2}{3}\ \text{inch per second}$$

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 1 of 2

5B

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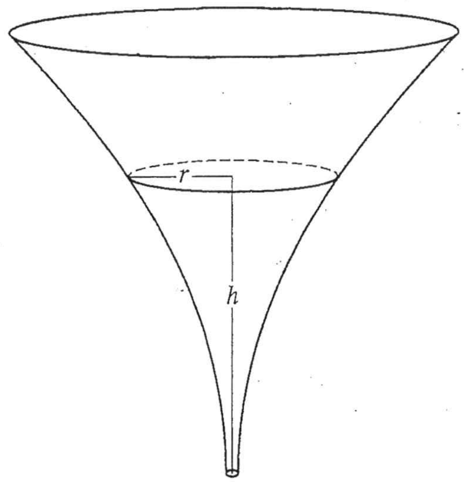

5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

$$\frac{1}{10}\int_0^{10} \frac{1}{20}(3+h^2)\,dh = \frac{1}{200}\int_0^{10}(3+h^2)\,dh$$

$$= \frac{1}{200}\left(3h + \frac{1}{3}h^3\right) + C\Big]_0^{10}$$

$$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - (0)\right) \text{ inches}$$

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Continue problem 5 on page 19.

(b) Find the volume of the funnel.

$$\pi \int_0^{10} \frac{1}{20}(3+h^2)\,dh = V$$

$$\frac{\pi}{20}\left(3h + \frac{1}{3}h^3\right) + C\Big]_0^{10} = \frac{\pi}{20}\left(30 + \frac{1000}{3}\right) \text{ cubic inches}$$

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\frac{dr}{dt} = \frac{1}{20}(2h)\left(\frac{dh}{dt}\right)$$

$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt} \qquad \frac{dh}{dt} = -\frac{10}{5(3)} \text{ inches per second}$$

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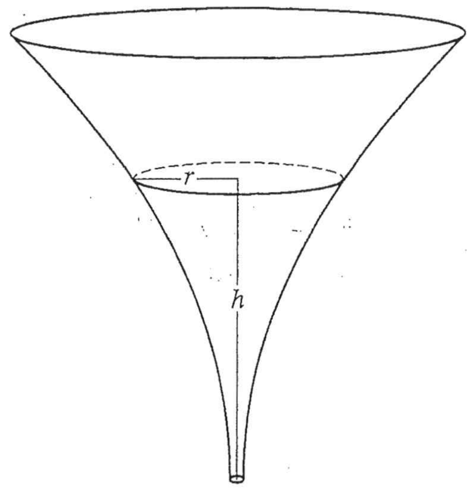
1 of 2

5C



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

$r = \frac{1}{20}(3+h^2)$. where $0 \le h \le 10$.

$\bar{r} = \frac{r(10) - r(0)}{10-0} = \frac{\frac{103}{20} - \frac{3}{20}}{10} = \frac{1}{2}$ (inches).

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Continue problem 5 on page 19.

2 of 2

5C

NO CALCULATOR ALLOWED

(b) Find the volume of the funnel.

$$V = \int_0^{10} \pi r^2 \, dh = \pi \int_0^{10} \left(\frac{3+h^2}{20}\right)^2 dh$$

$$= \pi \frac{1}{6h}\left(\frac{3+h^2}{20}\right)^3 \Big]_0^{10}$$

$$= \pi \cdot \frac{1}{60}\left(\frac{103}{20}\right)^2 - 0$$

$$= \frac{\pi}{60} \cdot \frac{103^2}{400}$$

$$= \frac{103^2 \pi}{24000} \text{ (inches}^3\text{)}$$

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$r = \frac{3}{20} + \frac{h^2}{20}$$

$$r' = \frac{h}{10} \cdot \frac{dh}{dt} = -\frac{1}{5}$$

$$h = 3$$

$$r' = \frac{3}{10} \cdot \frac{dh}{dt} = -\frac{1}{5}$$

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Question 5

Overview

In this problem students were presented with a funnel of height 10 inches and circular cross sections. At height h the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right),$ where r and h are measured in inches. In part (a) students needed to find the average value of the radius of the funnel. This required evaluating $\frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right)\,dh$ by finding an antiderivative. In part (b) students needed to find the volume of the funnel. By incorporating the fact that the cross sections are circular, the students were expected to set up and evaluate an integral of the form $\pi\int_0^{10} r^2\,dh = \pi\int_0^{10}\left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2\,dh.$ In part (c) students were given that the funnel contains liquid that is draining from the bottom. When the height of the liquid is 3 inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ in/sec. Students were expected to find the rate at which the height is changing at this instant. To solve this related rates problem, students needed to use $r = \frac{1}{20}\left(3 + h^2\right)$ and take the derivative with respect to t.

Sample: 5A
Score: 9

The response earned all 9 points.

Sample: 5B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the student does not present a correct integrand in the integral for volume and did not earn the first point. Without a correct integrand, the student was not eligible for the other points. In part (c) the student's work is correct.

Sample: 5C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student does not present an integral and did not earn the first point. Without presentation of an integral, the student was not eligible for the other points. In part (b) the student presents a correct integrand in the integral for volume and earned the first point. The student does not antidifferentiate correctly and did not earn the second point. The student was not eligible for the third point. In part (c) the student uses the chain rule correctly to find an equation relating $\frac{dh}{dt}$ to $\frac{dr}{dt}$ and earned the first 2 points. The student does not solve for $\frac{dh}{dt}$ and did not earn the third point.

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \leq \left|\frac{-1}{2^3 \cdot 3}(0.2)^3\right| = \frac{1}{3000} \leq 0.001$$

$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$1 + -\tfrac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2!}(x-1)^2 - \frac{2!}{2^3 \cdot 3!}(x-1)^3 \dots + (-1)^n \frac{(n-1)!}{2^n} \frac{(x-1)^n}{n!}$$

$$1 - \frac{x-1}{2} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{24} \quad \dots + (-1)^n \frac{(x-1)^n}{2^n \cdot n} + \dots$$

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{2^n \cdot n}$ converges over $x \in (-1, 3)$ (radius of convergence)

Endpoints:

$x = -1$ $\sum_{n=1}^{\infty} (-1)^n \frac{(-1-1)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} (-1)^n (-1)^n \frac{2^n}{2^n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, (harmonic series)

$x = 3$ $\sum_{n=1}^{\infty} (-1)^n \frac{(3-1)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2^n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alternating harmonic series converges by AST)

Interval of convergence: $x \in (-1, 3]$

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Continue problem 6 on page 21.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$1 - \frac{1.2-1}{2} + \frac{(1.2-1)^2}{8}$

$\frac{4}{800} = \frac{1}{200}$

$1 - .1 + \frac{.04}{8}$

$.9 + .005$

$.905 \approx f(1.2)$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

In Alternating taylor series $|\text{error}| \leq |\text{next neglected term}|$

$\frac{(1.2-1)^3}{24} \geq |\text{error}|$

$\frac{.008}{24} \geq |\text{error}|$

$\frac{.008}{24} \leq .001$

$|\text{error}| \leq .001$

approx is within .001 of exact value.

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NO CALCULATOR ALLOWED

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$1 - \frac{1}{2}(x-1) + \frac{1}{4}$

$f^2(1) = \frac{1}{4}$

$f^3(1) = -\frac{1}{4}$

$f^4(1) = \frac{3}{8}$

$1 - \frac{1}{2}(x-1) + \left(\frac{\frac{1}{4}(x-1)^2}{2!}\right) - \left(\frac{\frac{1}{4}(x-1)^3}{3!}\right) + \left(\frac{\frac{3}{8}(x-1)^4}{4!}\right)$

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| =$

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Continue problem 6 on page 21.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$$1 - \frac{1}{2}(1.2-1) + \frac{\frac{1}{4}(1.2-1)^2}{2!} \approx f(1.2)$$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$$\left| \frac{-\frac{1}{4}(1.2-1)^3}{3!} \right| < .001$$

error cannot be greater than the next term

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6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$f_{(4)}(x) = 1 - \tfrac{1}{2}(x-1) + \frac{\tfrac{1}{4}(x-1)^2}{2!} - \frac{\tfrac{1}{2}(x-1)^3}{3!}$$

$$f^{(2)}(1) = (-1)^2 \frac{(2-1)!}{2^2} = \tfrac{1}{4}$$

$$f^{(3)}(1) = (-1)^3 \frac{(3-1)!}{2^3} = \tfrac{4}{8} = \tfrac{1}{2}$$

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$$p(x) = \sum (-1)^n \frac{(n-1)!}{2^n} (x-1)^n$$

$$\lim_{n \to \infty} \left| \frac{(n)!}{2^{n+1}} (x-1)^{n+1} \right| \cdot \left| \frac{2^n}{(n-1)!} \left(\frac{1}{(x-1)^n} \right) \right|$$

$$\lim_{n \to \infty} \left| \frac{n(x-1)}{2} \right| = \infty$$

$$(-\infty, \infty)$$

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Continue problem 6 on page 21.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$f_{(3)}(x) = 1 - \frac{1}{2}(x-1) + \frac{\frac{1}{4}(x-1)^2}{2!}$

$\frac{0.04}{8}$

$f(1.2) = 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2$

$f(1.2) = 1 - 0.1 + 0.04/8$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$|f(1.2) - 1.2| < 0.001$

$\left|1 - 0.1 + \frac{0.04}{8} - 1.2\right| < 0.001$

$\left|-0.3 + \frac{0.04}{8}\right| < 0.001$

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Question 6

Overview

In this problem students were presented with a function f that has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. The values of f, the first derivative of f, and nth derivatives of f at $x = 1$ are given. In part (a) students needed to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$. In part (b) students were given the radius of convergence and asked to find the interval of convergence. Since the series is centered about $x = 1$ with a radius of convergence of 2, students were expected to determine if the series converges at the endpoints $x = -1$ and $x = 3$. Students needed to use knowledge of the harmonic series and alternating harmonic series. In part (c) students were expected to use the Taylor series found in part (a) to represent $f(1.2)$ as an alternating series, and then use the first three nonzero terms of the alternating series to approximate $f(1.2)$. In part (d) students were expected to show that the approximation from part (c) is within 0.001 of the exact value of $f(1.2)$. The error of this approximation is bounded by the magnitude of the fourth term of the series for $f(1.2)$.

Sample: 6A
Score: 9

The response earned all 9 points. In part (a) the student presents the first four nonzero terms and a simplified version of the general term. In part (b) the student identifies the endpoints, substitutes the values into the series, correctly simplifies and identifies each series for the analysis, and presents the correct interval of convergence. In part (c) the student correctly substitutes into the first three terms and correctly evaluates to obtain the approximation. In part (d) the student correctly uses the next term as a bound on the error and compares the value to 0.001.

Sample: 6B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student correctly presents the first four nonzero terms of the series to earn the first 3 points. The student does not present a general term. In part (b) the student does not identify the correct endpoints and did not earn either point. Note that with a given radius of convergence R, it is not necessary for the student to use the ratio test to determine an interval for consideration. The student could have started this problem by considering $(1 - R, 1 + R)$. In part (c) the student's work is correct. The student uses the first three terms of the series to produce the correct approximation for $f(1.2)$. In part (d) the student correctly uses the next term as a bound on the error and compares the value to 0.001.

Sample: 6C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student correctly finds the first three nonzero terms of the series and earned the first 2 points. The student has an incorrect coefficient for the fourth term. The student does not present a general term. In part (b) the student does not identify the correct endpoints and did not earn either point. In part (c) the student's work is correct. The student uses the first three terms of the series to produce the correct approximation for $f(1.2)$. In part (d) the student does not present a correct error form.